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ON THE ASSOCIATIVE HOMOTOPY LIE ALGEBRAS AND THE WRONSKIANS

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Abstract. Representations of the Schlessinger–Stasheff's associative homotopy Lie algebras in the spaces of higher–order differential operators are analyzed. The W-transformations of chiral embeddings, related with the Toda equations, of complex curves into the Kähler manifolds are shown to be endowed with the homotopy Lie algebra structures. Extensions of the Wronskian determinants that preserve the properties of the Schlessinger–Stasheff algebras are constructed for the case of $n \geq 1$ independent variables.

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Introduction. Recently, the attention of the mathematical physics community has been drawn to the N-ary Lie algebra structures, i.e., the N-linear skew–symmetric brackets that satisfy an analog of the Jacobi identity, and to the N-field dynamics problems, e.g., N-ary objects similar to the Poisson manifolds. There are several concepts ([5, 7, 13, 20, 25, 26]) how the Jacobi identity should be generalized, each of them having its own interpretation (e.g., [24]) and applications ([4]). Recently, a unifying

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approach was proposed in [28], treating the known cases as the special ones within the 3-parameter family of the identities.

Up to our present knowledge, the papers [7, 26], issued in 1985, were the first works on the topic. In [7], V. T. Filippov considered an N-linear skew–symmetric bracket ∇, defined on a vector space $\mathcal{A}$, that satisfied an analog of the Jacobi identity:
(I.1)

$$\nabla\left(a_1,\ldots,a_{N-1},\nabla(b_1,\ldots,b_N)\right)=\sum_{i=1}^{N}\nabla\left(b_1,\ldots,b_{i-1},\nabla(a_1,\ldots,a_{N-1},b_i),b_{i+1},\ldots,b_N\right),$$

where $a_i,\ b_j\in\mathcal{A}$; then, $\mathcal{A}$ is called an N-Lie algebra. The motivation of (I.1) to appear is quite understandable: the adjoint representation $a\mapsto\nabla(a_1,\ \ldots,\ a_{N-1},\ a)$ is a derivation for any $a_i\in\mathcal{A}$. The standard constructions of the Lie algebra theory for the N-Lie algebras were introduced in [15].

The Nambu mechanics is an example of the N-Poisson dynamics assigned to identity (I.1); in [24], the standard binary Poisson bracket was replaced by the ternary ($N=3$) one:

$$\nabla(f_1,\ldots,f_N)=\det\left\|\frac{\partial f_i}{\partial x^j}\right\|,$$

where $\mathcal{A}=C^\infty(\mathbb{R}^N)$ and the r.h.s. contains the Jacobi determinant; nevertheless, the fact that Eq. (I.1) holds for this ∇ was not noticed until [25]. Then, L. Takhtajan ([27]) developed the concept of the Nambu–Poisson manifolds for $N\geq 2$.

The second natural generalization of the ordinary Jacobi identity is

$$\sum_{\sigma\in S^N_{2N-1}}(-1)^\sigma\Delta(\Delta(a_{\sigma(1)},\ldots,a_{\sigma(N)}),a_{\sigma(N+1)},\ldots,a_{\sigma(2N-1)})=0, \tag{I.2}$$

where $a_i\in\mathcal{A}$ and $S^N_{2N-1}=\{\sigma\in S_{2N-1}\mid\sigma(1)<\cdots<\sigma(N),\ \sigma(N+1)<\cdots<\sigma(2N-1)\}$ is the set of the unshuffle permutations. These brackets Δ are named the homotopy N-Lie algebra structures ([26]) and are closely related with the SH–algebras ([2, 20]). Also, these algebras and their Koszul cohomologies were studied in [13]; their Hochschild cohomologies were considered in [23].

The N-Poisson manifolds associated with Jacobi's identity (I.2) were introduced in [1]: an N-vector field V is an N-Poisson structure if the equation $[\![V,V]\!]^{\mathrm{S}}=0$ for the Schouten bracket holds (see page 4 for definitions).

The properties of the Filippov's N-Lie and the Schlessinger–Stasheff's homotopy N-Lie algebras are quite different; really, they appear in the 3-parametric scheme ([28]) as the opposite cases: (N, $N-1$, 0) and (N, 0, 0), respectively (see Definition 2 below). Further discussion on the topic is found in [28].

In the sequel, we analyze the properties of *associative* homotopy Lie algebras and their representations in higher order differential operators. Namely, we relate the corresponding structures with the Wronskian determinants, point out N-ary analogues of vector fields on smooth manifolds, and construct a definition of the Wronskian for $n\geq 1$ independent variables which is correlated with the structures defined by Eq. (I.2). Also, we prove that the W-transformations of the W-surfaces ([9]–[11]) in the Kähler manifolds are endowed with the homotopy Lie algebra structures. We use the jet

bundles language [3]; our approach is aimed to contribute the study of related aspects in the cohomological algebra and the field theory. The exposition patterns upon [17, Chapter V].

The paper is organized as follows.

In Section 1, we introduce the main algebraic concept of the homotopy N-Lie algebras. We fix notation and define the Richardson–Nijenhuis bracket $[\![\cdot,\cdot]\!]^{\mathrm{RN}}$, the homotopy N-Jacobi identities $[\![\Delta,\Delta]\!]^{\mathrm{RN}} = 0$ for N-linear skew–symmetric operators Δ, and the Hochschild and Koszul cohomologies. Next, we illustrate the definitions by two finite–dimensional homotopy Lie algebras which are analogues the Lie algebra $\mathfrak{sl}_2(\Bbbk)$.

In Section 2, we consider representations of the homotopy Lie algebra structures in the higher order differential operators over $\Bbbk$. Analyzing the properties of the corresponding N-linear skew–symmetric bracket, we point out higher–order generalizations of vector fields and thus explain the property $\partial^{\vec{\imath}}[\partial^{\vec{\imath}}] = 0$ of the Wronskian determinants $\partial^{\vec{\imath}} = \mathbf{1} \wedge \partial \wedge \cdots \wedge \partial^{N-1}$ to provide the homotopy Lie algebra structures for even Ns; also, we calculate the structural constants of these algebras in terms of the Vandermonde determinants. Next, we relate the multilinear homotopy structures with the Toda equations ([21]). The latter are known to be the compatibility condition in the W-geometry ([9]–[11]) of chiral embeddings of complex curves into the Kähler manifolds, while higher order differential operators, endowed with the homotopy Lie algebra structures, are admissible deformations of these embeddings.

In Section 3, we construct analogues $D^{\vec{\sigma}} = D_{\sigma_1} \wedge \cdots \wedge D_{\sigma_N}$ of the Wronskian determinants $\partial^{\vec{\imath}}$ for the case of $n \geq 1$ independent variables x^1, …, x^n, such that the $\binom{n+k}{n}$-Jacobi identities $D^{\vec{\sigma}}[D^{\vec{\sigma}}] = 0$ are preserved for any integers n, $k \geq 1$.

1. Preliminaries: the algebraic concept

1.1. **Basic definitions and facts.** First let us introduce some notation. Let $\mathcal{A}$ be an algebra over the field $\Bbbk$ such that $\operatorname{char}\Bbbk = 0$ and let ∂ be a derivation of $\mathcal{A}$. As an illustrative example, one can think $\mathcal{A}$ to be the algebra of smooth functions $f\colon M \to \mathbb{R}$ on a smooth real manifold M.

Let $S^k_m \subset S_m$ be the *unshuffle permutations* such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(m)$ for any $\sigma \in S^k_m$. Let $\Delta \in \operatorname{Hom}_{\Bbbk}(\bigwedge^k \mathcal{A}, \mathcal{A})$ be a homomorphism and take arbitrary $a_j \in \mathcal{A}$, $1 \leq j \leq k$; suppose $1 \leq l \leq k$. The *inner product* $\Delta_{a_1,\ldots,a_m} \in \operatorname{Hom}(\bigwedge^{k-m} \mathcal{A}, \mathcal{A})$ is the operator defined by the rule

$$\Delta_{a_1,\ldots,a_m}(a_{m+1},\ldots,a_k) \stackrel{\mathrm{def}}{=} \Delta(a_1,\ldots,a_k).$$

Definition 1 ([8, 28])**.** Let $\Delta \in \operatorname{Hom}(\bigwedge^k \mathcal{A}, \mathcal{A})$ and $\nabla \in \operatorname{Hom}(\bigwedge^l \mathcal{A}, \mathcal{A})$ be two operators; by definition, the *exterior multiplication* $\wedge$ in $\operatorname{Hom}(\bigwedge^* \mathcal{A}, \mathcal{A})$ is

$$(\Delta \wedge \nabla)(a_1,\ldots,a_{k+l}) = \sum_{\sigma \in S^k_{k+l}} (-1)^\sigma\, \Delta(a_{\sigma(1)},\ldots,a_{\sigma(k)}) \cdot \nabla(a_{\sigma(k+1)},\ldots,a_{\sigma(k+l)})$$

for any $a_1,\ldots,a_{k+l} \in \mathcal{A}$.

Example 1. The exterior multiplication $\wedge$ on the space of higher–order differential operators that act on the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on M defines the Wronskian determinants $W^{0,1,\ldots,N+1} = \partial^0 \wedge \ldots \wedge \partial^{N+1}$. In this paper, we also consider

generalized Wronskians $W^{\vec{\imath}} = \partial^{i_1} \wedge \ldots \wedge \partial^{i_N} \in \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$, where ∂ is a derivation of $\mathcal{A}$. Let a multiindex $\vec{\imath} \in \mathbb{Z}_+^N$ be such that $0 \leq i_1 < \ldots < i_N$. By $\mathrm{Hom}_t(\bigwedge^N \mathcal{A}, \mathcal{A})$ we denote the linear span of the generalized Wronskians $W^{\vec{\imath}}$ such that $|\vec{\imath}| \equiv \sum_j i_j = t$. By definition, put $|W^{\vec{\imath}}| = |\vec{\imath}|$. In Section 3, we construct generalizations of the Wronskian determinants for analytic functions $\Bbbk[[x^1, \ldots, x^n]]$ by using Definition 1.

Next, let $\Delta \in \mathrm{Hom}(\bigwedge^k \mathcal{A}, \mathcal{A})$ and $\nabla \in \mathrm{Hom}(\bigwedge^l \mathcal{A}, \mathcal{A})$. By $\Delta[\nabla] \in \mathrm{Hom}(\bigwedge^{k+l-1} \mathcal{A}, \mathcal{A})$ we denote the *action* $\Delta[\cdot]\colon \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A}) \to \mathrm{Hom}(\bigwedge^{N+k-1} \mathcal{A}, \mathcal{A})$ of Δ on ∇:

$$\Delta[\nabla](a_1, \ldots, a_{k+l-1}) \stackrel{\mathrm{def}}{=} \sum_{\sigma \in S^l_{k+l-1}} (-1)^\sigma \, \Delta(\nabla(a_{\sigma(1)}, \ldots, a_{\sigma(l)}), a_{\sigma(l+1)}, \ldots, a_{\sigma(k+l-1)}), \tag{1}$$

where $a_j \in \mathcal{A}$. By $[\![\Delta, \nabla]\!]^{\mathrm{RN}} \in \mathrm{Hom}(\bigwedge^{k+l-1} \mathcal{A}, \mathcal{A})$ we denote the *Richardson–Nijenhuis bracket* of Δ and ∇:

$$[\![\Delta, \nabla]\!]^{\mathrm{RN}} \stackrel{\mathrm{def}}{=} \Delta[\nabla] - (-1)^{(k-1)(l-1)} \nabla[\Delta]. \tag{2}$$

Definition 2 ([28])**.** Choose integers N, k, and r such that $0 \leq r \leq k < N$, and let $a_1, \ldots, a_r, b_1, \ldots, b_k \in \mathcal{A}$. The skew-symmetric map $\Delta \in \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ is said to determine the Lie algebra structure of the type (N, k, r) on the $\Bbbk$-vector space $\mathcal{A}$ if Δ satisfies the (N, k, r)-*Jacobi identity*

$$[\![\Delta_{a_1, \ldots, a_r}, \Delta_{b_1, \ldots, b_k}]\!]^{\mathrm{RN}} = 0 \tag{3}$$

for any $\vec{a}$ and $\vec{b}$. By $\mathrm{Lie}^{(N,k,r)}(\mathcal{A})$ we denote the set of all type (N, k, r) structures $\Delta \in \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ on $\mathcal{A}$.

The structure $\Delta \in \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ is called a *multi-derivation* if the Leibnitz rule

$$\Delta(ab, a_2, \ldots, a_N) = a \, \Delta(b, a_2, \ldots, a_N) + \Delta(a, a_2, \ldots, a_N) \, b$$

is valid for any a, b, $a_j \in \mathcal{A}$.

Example 2 (Filippov's N-Lie algebras). Consider the family $\mathrm{Lie}^{(N,N-1,0)}$ for integer $N \geq 2$. The N-Jacobi identity is (I.1), meaning that the adjoint representation for these algebras is a derivation. This case is a natural generalisation of the Poisson theory ([27]).

Remark 1 ([28]). We have

$$\mathrm{Lie}^{(N,0,0)}(\mathcal{A}) = \mathrm{Lie}^{(N,1,0)}(\mathcal{A}) \tag{4}$$

for any even N; this is a typical instance of the heredity structures. Indeed, the following two conditions are equivalent:

$$[\![\Delta, \Delta]\!]^{\mathrm{RN}} = 0 \Leftrightarrow [\![\Delta, \Delta]\!]^{\mathrm{RN}}{}_a = -2 \, [\![\Delta, \Delta_a]\!]^{\mathrm{RN}} = 0 \qquad \forall a \in \mathcal{A},$$

owing to Corollary 1.1 in [28]:

$$[\![\Delta, \Delta]\!]^{\mathrm{RN}}{}_a = (-1)^{N-1} [\![\Delta, \Delta_a]\!]^{\mathrm{RN}} + [\![\Delta_a, \Delta]\!]^{\mathrm{RN}}.$$

Finally, $[\![\Delta, \Delta_a]\!]^{\mathrm{RN}} = 0$ for any $a \in \mathcal{A}$.

Let $\Delta \in \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ be an N-linear skew-symmetric bracket: $\Delta(a_{\Sigma(1)}, \ldots, a_{\Sigma(N)}) = (-1)^\Sigma \Delta(a_1, \ldots, a_N)$ for any rearrangement $\Sigma \in S_N$.

Definition 3. The algebra $\mathcal{A}$ is the *homotopy N-Lie* algebra, or the *Schlessinger–Stasheff N-algebra*, if the N-Jacobi identity

$$\Delta[\Delta] = 0 \tag{5}$$

holds.

In coordinates, the N-Jacobi identity is

$$\sum_{\sigma\in S^N_{2N-1}} (-1)^\sigma \Delta(\Delta(a_{\sigma(1)},\dots,a_{\sigma(N)}), a_{\sigma(N+1)},\dots,a_{\sigma(2N-1)}) = 0 \tag{6}$$

for any $a_j \in \mathcal{A}$, $1 \le j \le 2N-1$. Generally, the number of summands in (6) is $\binom{2N-1}{N-1} = \binom{2N-1}{N}$, see [18].

The Jacobi identity of the type $(N,0,0)$ $[\![\Delta,\Delta]\!]^{\mathrm{RN}} = 2\Delta[\Delta] = 0$ implies Eq. (6) for any even N. If N is odd, then the expression

$$[\![\Delta,\Delta]\!]^{\mathrm{RN}} = 0 \tag{7}$$

is trivial and we consider Eq. (5) separately from condition (3). In the sequel, we study the Jacobi identity (6) of the form (5), where $\Delta \in \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$.

1.2. **The Hochschild and the Koszul cohomologies.** The graded Jacobi identity for the Richardson–Nijenhuis bracket provides the Hochschild d_Δ-cohomologies on $\mathrm{Hom}(\bigwedge^* \mathcal{A}, \mathcal{A})$ for $\Delta \in \mathrm{Hom}(\bigwedge^k \mathcal{A}, \mathcal{A})$, where k is even:

Proposition 1 ([28])**.** *The Richardson–Nijenhuis bracket satisfies the graded Jacobi identity*

$$[\![\Delta,[\![\nabla,\Box]\!]^{\mathrm{RN}}]\!]^{\mathrm{RN}} = [\![[\![\Delta,\nabla]\!]^{\mathrm{RN}},\Box]\!]^{\mathrm{RN}} + (-1)^{(\Delta-1)(\nabla-1)}[\![\nabla,[\![\Delta,\Box]\!]^{\mathrm{RN}}]\!]^{\mathrm{RN}}. \tag{8}$$

Corollary 2. Let k be an even natural number and an operator $\Delta \in \mathrm{Hom}(\bigwedge^k \mathcal{A}, \mathcal{A})$ be such that $[\![\Delta,\Delta]\!]^{\mathrm{RN}} = 0$; then the operator $\mathrm{d}_\Delta \equiv [\![\Delta,\cdot]\!]^{\mathrm{RN}}$ is a differential: $\mathrm{d}_\Delta^2 = 0$.

The cohomologies w.r.t. the differential d_Δ are called the Hochschild d_Δ-cohomologies of the space $\mathrm{Hom}_{\Bbbk}(\bigwedge^* \mathcal{A}, \mathcal{A})$.

Remark 2. The approach under study is closely related with the algebraic Hamiltonian formalism in the geometry of partial differential equations ([3, 16]): a bi-vector A endowes the space of the Hamiltonians with the Lie algebra structure iff its Schouten bracket with itself satisfies the equation $[\![A,A]\!]^{\mathrm{S}} = 0$; from the Jacobi identity similar to Eq. (8) it follows that the operator $\mathrm{d}_A = [\![A,\cdot]\!]^{\mathrm{S}}$ is a differential and therefore defines the Hamiltonian complex whose cohomologies are called the Poisson cohomologies. We note that the operator $W^{0,1} = \mathbf{1} \wedge d/dx$, which is studied in Section 2.2, is the first Hamiltonian structure for the Korteweg–de Vries equation.

Let the bracket $\Delta \in \mathrm{Hom}(\bigwedge^k \mathcal{A}, \mathcal{A})$ satisfy the homotopy k-Jacobi identity $\Delta[\Delta] = 0$. By ∂_Δ denote the linear map $\partial_\Delta \in \mathrm{Hom}(\bigwedge^r \mathcal{A}, \bigwedge^{r-k+1} \mathcal{A})$ such that

(1) $\partial_\Delta\Big|_{\bigwedge^r \mathcal{A}} = 0$ if $r < k$;
(2) $\partial_\Delta(a_1 \wedge \ldots \wedge a_r) = \sum\limits_{\sigma\in S^k_r} (-1)^\sigma \Delta[a_{\sigma(1)},\dots,a_{\sigma(k)}] \wedge a_{\sigma(k+1)} \wedge \ldots \wedge a_{\sigma(r)}$ otherwise.

We obtain the Koszul ∂_Δ-cohomologies for the algebra $\bigwedge^* \mathcal{A} = \bigoplus_{r=2}^{\infty} \bigwedge^r \mathcal{A}$ over the algebra $\mathcal{A}$ owing to

Proposition 3 ([13]). *The operator* $\partial_\Delta \colon \bigwedge^* \mathcal{A} \to \bigwedge^* \mathcal{A}$ *is a differential:* $\partial_\Delta^2 = 0$.

By $H^*_\Delta(\mathcal{A})$ we denote the Koszul ∂_Δ-cohomologies w.r.t. the differential ∂_Δ. For $N = 2$, the Koszul cohomologies of the Lie algebra of vector fields on the circumpherence $\mathbb{S}^1$ were obtained in [8]. For $N \geq 2$, the Koszul ∂_Δ-cohomologies of free algebras were found in [13].

1.3. **Examples of the homotopy Lie algebras.** One should notice that algebraic structures (5) have a remarkable geometric motivation to exist. Namely, we have

Example 3 ([13]). Let $\mathcal{A} = \Bbbk^r$ be a $\Bbbk$-linear space of arbitrary dimension r and $\Delta \colon \bigwedge^N \mathcal{A} \to \mathcal{A}$ be a skew-symmetric linear mapping of $\mathcal{A}$. If $\dim \mathcal{A} < 2N - 1$, then the identity (6) holds for Δ.

Proof. We maximize the number of summands in (6) in order to note its skew-symmetry w.r.t. the transpositions $a_j \mapsto a_{\Sigma(j)}$, $\Sigma \in S_{2N-1}$. The l.h.s. of Jacobi identity (6) equals

$$\frac{1}{N!(N-1)!} \sum_{\sigma \in S_{2N-1}} (-1)^\sigma \Delta(\Delta(a_{\sigma(1)}, \ldots, a_{\sigma(N)}), a_{\sigma(N+1)}, \ldots, a_{\sigma(2N-1)}), \tag{9}$$

where *all* elements $\sigma \in S_{2N-1}$ are taken into consideration; see [13, 23]. Expression (9) is skew-symmetric w.r.t. any rearrangement Σ of the elements $a_j \in \vec{a}$:

$$\begin{aligned} (-1)^\Sigma \sum_{\sigma \in S_{2N-1}} (-1)^\sigma \Delta(\Delta(a_{(\sigma\circ\Sigma)(1)}, \ldots, a_{(\sigma\circ\Sigma)(N)}), a_{(\sigma\circ\Sigma)(N+1)}, \ldots, a_{(\sigma\circ\Sigma)(2N-1)}) = \\ = \sum_{\sigma \in S_{2N-1}} (-1)^\sigma \Delta(\Delta(a_{\sigma(1)}, \ldots, a_{\sigma(N)}), a_{\sigma(N+1)}, \ldots, a_{\sigma(2N-1)}). \end{aligned} \tag{10}$$

Consequently, the l.h.s in (9) is skew-symmetric also and we obtain a $(2N-1)$-linear skew-symmetric operator acting on the vector space of smaller dimension. Hence, if $\dim \mathcal{A} < 2N - 1 =$ the number of arguments $\sharp \vec{a}$, then (6) holds. □

• The case $\dim \mathcal{A}_1 = N + 1$, $[a_0, \ldots, \widehat{a_j}, \ldots, a_N] = (-1)^j \cdot a_j$ is well-known to be the cross-product in $\Bbbk^{N+1}$. For $\Bbbk = \mathbb{R}$ and $N = 2$, we have the Lie algebra $\mathcal{A} \simeq \mathfrak{so}(3)$.
• We claim that the algebra $\mathcal{A}_2$ of dimension $\dim \mathcal{A}_2 = N + 1$, defined by the relations

$$[a_0, \ldots, \widehat{a_j}, \ldots, a_N] = a_{N-j}, \quad 0 \leq j \leq N, \tag{11}$$

admits a representation in the space of polynomials $\Bbbk_N[x]$ such that its structure $[\ldots]$ is defined by the Wronskian determinants of scalar fields (smooth functions of one argument). The algebra $\mathcal{A}_2$ is considered in the next subsection.

1.4. **The polynomials.** In this section, we construct finite-dimensional homotopy N-Lie generalizations of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$. Our starting point is the following

Example 4 ([18]). The polynomials $\Bbbk_2[x] = \{\alpha x^2 + \beta x + \gamma \mid \alpha, \beta, \gamma \in \Bbbk\}$ of degree 2 form a Lie algebra isomorphic to $\mathfrak{sl}_2(\Bbbk)$. The latter is generated by three elements $\langle e, h, f\rangle$ that satisfy the relations

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Consider the basis $\langle 1, -2x, -x^2\rangle$ and choose the Wronskian determinant for the bracket on $\Bbbk_2[x]$:

$$[-2x, 1] = 2, \qquad [-2x, -x^2] = 2x^2, \qquad [1, -x^2] = -2x,$$

whence the representation $\rho\colon \mathfrak{sl}_2(\Bbbk) \to \Bbbk_2[x]$ is

$$\rho(e) = 1, \qquad \rho(h) = -2x, \qquad \text{and} \quad \rho(f) = -x^2. \tag{12}$$

Consider the space $\Bbbk_N[x] \ni a_j$ of polynomials a_j of degree not greater than N; on this space, there is the N-linear skew-symmetric bracket

$$[a_1, \dots, a_N] = W(a_1, \dots, a_N), \tag{13}$$

where W denotes the Wronskian determinant. Since N-ary bracket (13) is N-linear, we consider monomials $\mathrm{const}\cdot x^k$ only. We choose $\{a_j^0\} = \{x^k\}$ or $\{a_j^0\} = \{x^k/k!\}$, where $0 \le k \le N$ and $1 \le j \le 2N-1$, for standard basis in $\Bbbk_N[x]$. The powers $x^0, \dots, x^N$ and n independent variables $x^1, \dots, x^n$ introduced in Section 3 are never mixed, and the notation is absolutely clear from the context. Exact choice of the basis depends on the situation: the monomials x^k are used to demonstrate the presence or absence of certain degrees in N-linear bracket (13) and the monomials $x^k/k!$ are convenient in calculations since they are closed w.r.t. the derivations (and the Wronskian determinants as well). Indeed, we have

Theorem 4 ([19]). *Let $0 \le k \le N$; then the relation*

$$W\left(1, \dots, \widehat{\frac{x^k}{k!}}, \dots, \frac{x^N}{N!}\right) = \frac{x^{N-k}}{(N-k)!} \tag{14}$$

holds.

Proof. We have

$$W\left(1, \dots, \widehat{\frac{x^k}{k!}}, \dots, \frac{x^N}{N!}\right) = W\left(1, \dots, \frac{x^{k-1}}{(k-1)!}\right) \cdot W\left(x, \dots, \frac{x^{N-k}}{(N-k)!}\right), \tag{15}$$

where the first factor in the r.h.s. of (15) equals 1 and has the degree 0. Denote the second factor, the determinant of the $(N-k)\times(N-k)$ matrix, by W_m, $m \equiv N-k$. We claim that W_m is a monomial: $\deg W_m = m$, and prove this fact by induction on $m \equiv N-k$. For $m = 1$, $\deg\det(x) = 1 = m$. Let $m > 1$; the decomposition of W_m w.r.t. the last row gives

$$W_m = W\left(x, \dots, \frac{x^m}{m!}\right) = x \cdot W\left(x, \dots, \frac{x^{m-1}}{(m-1)!}\right) - W\left(x, \dots, \frac{x^{m-2}}{(m-2)!}, \frac{x^m}{m!}\right), \tag{16}$$

where the degree of the first Wronskian in r.h.s. of (16) is $m-1$ by the inductive assumption. Again, decompose the second Wronskian in r.h.s. of (16) w.r.t. the last

row and proceed iteratively by using the induction hypothesis. We obtain the recurrence relation

$$W_m = \sum_{l=1}^{m-1} W_{m-l} \cdot (-1)^{l+1} \frac{x^l}{l!} - (-1)^m \frac{x^m}{m!}, \qquad m \geq 1, \tag{17}$$

whence $\deg W_m = m$. We see that the initial Wronskian (15) is a monomial itself of degree $m = N - k$ with yet unknown coefficient.

Now we calculate the coefficient $W_m(x)/x^m \in \Bbbk$ in the Wronskian determinant (15). Consider the generating function

$$f(x) \equiv \sum_{m=1}^{\infty} W_m(x) \tag{18}$$

such that

$$W_m(x) = \frac{x^m}{m!} \frac{d^m f}{dx^m}(0), \qquad 1 \leq m \in \mathbb{N}.$$

Recall that $\exp(x) \equiv \sum_{m=0}^{\infty} x^m/m!$; treating (18) as the formal sum of equations (17), we have

$$f(x) = f(x) \cdot (\exp(-x) - 1) - \exp(-x) + 1,$$

whence

$$f(x) = \exp(x) - 1. \tag{19}$$

Hence the required coefficient equals $1/m!$. The proof is complete. □

We have shown that the polynomials $\Bbbk_N[x]$ are closed w.r.t. the Wronskian determinant, and we know that any N-linear skew-symmetric bracket Δ on $\Bbbk^{N+1}$ satisfies $\Delta[\Delta] = 0$. Therefore, the statement that the polynomials $\Bbbk_N[x]$ of degree not greater than N form the homotopy N-Lie algebra with N-linear skew-symmetric bracket (13) for any integer $N \geq 2$ is quite obvious. Nevertheless, in the sequel we show that the Wronskian $W^{0,1,\ldots,N-1} \in \mathrm{Hom}(\bigwedge^N \Bbbk_N[x], \Bbbk_N[x])$ is the restriction of a *nontrivial* homotopy N-Lie bracket that lies in $\mathrm{Hom}(\bigwedge^N \Bbbk[[x]], \Bbbk[[x]])$. Also, the dimension n of the base $\Bbbk \equiv \Bbbk^1 \ni x$ equals 1. In Section 3, we generalize the concept to the case $x \in \Bbbk^n$, where integer $n \geq 1$ is arbitrary.

2. The associative homotopy Lie algebras

Another natural example of the homotopy Lie algebras is given by

Proposition 5 ([4, 13])**.** *Let $\mathcal{A}$ be an associative algebra and let N be even; by definition, put*[1]

$$[a_1, \ldots, a_N] \stackrel{\text{def}}{=} \sum_{\sigma \in S_N} (-1)^\sigma \cdot a_{\sigma(1)} \circ \cdots \circ a_{\sigma(N)}. \tag{20}$$

Then $\mathcal{A}$ is a homotopy Lie algebra w.r.t. this bracket.

[1]Note that the permutations $\sigma \in S_N$ provide the *direct* left action on $\bigotimes^N \mathcal{A}$, contrary to the inverse action in [14, §II.2.6]. By definition, $\sigma(j)$ is the index of the object in an initial ordered set placed onto jth position after the left action of a permutation σ.

Proof ([13]). The crucial idea is using (9) and (10). Let a_1, …, a_{2N-1} lie in $\mathcal{A}$ and $\sigma \in S_{2N-1}$ be a permutation. In order to compute the coefficient of $a_{\sigma(1)} \circ \cdots \circ a_{\sigma(2N-1)}$ in (6) and prove it to be trivial, it is enough to do that for $\alpha = a_1 \circ \cdots \circ a_{2N-1}$ in (6) due to (10) and (9), successively.

Now we use the assumption $N \equiv 0 \mod 2$. The product α is met N times in (6) in the summands β_j, $1 \le j \le N$:

$$\beta_j = (-1)^{N(j-1)} \, [[a_j, \ldots, a_{N+j-1}], a_1, \ldots, a_{j-1}, a_{N+j}, \ldots, a_{2N-1}]. \tag{21}$$

The coefficient of α in β_j equals $(-1)^{j-1}$ and hence the coefficient of α in (6) is

$$\sum_{j=1}^{N} (-1)^{j-1} = 0.$$

The proof is complete. □

From the proof of Proposition 5 we see that the main obstacle for bracket (20) to provide the homotopy Lie algebra structures for odd Ns are the signs within (1), (20), and in the Richardson–Nijenhuis bracket (2) that defines the Jacobi identity as the cohomological conditions $\mathrm{d}_\Delta^2 = 0$, see Proposition 1. Namely, we have

Proposition 6 ([4]). *Let the subscript i at the bracket's* (20) *symbol Δ_i denote its number of arguments*: $\Delta_i \in \mathrm{Hom}_{\Bbbk}(\bigwedge^i \mathcal{A}, \mathcal{A})$, *and let k and ℓ be arbitrary integers. Then the following identities hold*:

$$\Delta_{2k}[\Delta_{2\ell}] = 0, \tag{22a}$$

$$\Delta_{2k+1}[\Delta_{2\ell}] = \Delta_{2k+2\ell}, \tag{22b}$$

$$\Delta_{k}[\Delta_{2\ell+1}] = k \cdot \Delta_{2\ell+k}. \tag{22c}$$

Proof. The proof of (22a) repeats the reasoning in (21) literally. For (22b), we note that the last summand β_{2k+1} is not compensated. For (22c), the summand $\alpha = a_1 \circ \cdots \circ a_{2\ell+k}$ acquires the coefficient

$$\sum_{j=1}^{k} (-1)^{(2\ell+1)(j-1)} \cdot (-1)^{j-1} = k.$$

This completes the proof. □

2.1. **On representations in differential operators.** In this section, the field $\Bbbk$ is the complex field $\mathbb{C}$: $\Bbbk = \mathbb{C}$, and z is the holomorphic coordinate in $\mathbb{C}$.

By $\mathcal{O}(\mathbb{C})$ we denote the algebra of the Laurent series over $\mathbb{C}$. Consider the associative algebra $\mathrm{Diff}_*(\mathcal{O}(\mathbb{C}), \mathcal{O}(\mathbb{C}))$ of holomorphic differential operators

$$\nabla_{\vec{w}} = \sum_{j=0}^{p} w_j(z) \cdot \partial^j. \tag{23}$$

We claim that for any p the algebra $\mathrm{Diff}_*(\mathcal{O}(\mathbb{C}), \mathcal{O}(\mathbb{C}))$ is a homotopy Lie algebra that contains a homotopy $2p$-Lie subalgebra defined by a skew–symmetric bracket of $2p$ arguments. Recall that for $N = 2$ vector fields compose a subalgebra in the space of differential operators of arbitrary orders.

Let $a_j \in \mathrm{Diff}_*(\mathbb{C})$ be $a_j = w_j(z)\,\partial^{k_j}$ for $1 \leq j \leq N$. Similar to (20), put

$$[w_1 \cdot \partial^{k_1}, \ldots, w_N \cdot \partial^{k_N}] \stackrel{\mathrm{def}}{=} \sum_{\sigma \in S_N} (-1)^\sigma\, w_{\sigma(1)}\, \partial^{k_{\sigma(1)}} \circ \cdots \circ w_{\sigma(N)} \cdot \partial^{k_{\sigma(N)}}. \tag{24}$$

This bracket is N-linear over $\mathbb{C}$ and is skew-symmetric w.r.t. permutations of its arguments.

First, we count derivatives: Consider the special case $k_j \equiv p = \mathrm{const}$ for all j and solve the equation

$$Np = \frac{N(N-1)}{2} + p \tag{25}$$

for p: $p = N/2$; note that $N(N-1)/2 = |W^{0,1,\ldots,N-1}|$.

Further on, we restrict ourselves to the case $N \equiv 0 \mod 2$; it turns out that for odd Ns we need to consider half-integer powers of the derivation ∂: ∂^0, $\partial^{1/2}$, ∂, $\partial^{3/2}$, ….

Theorem 7. *Let N be even and $w_j \in \mathcal{O}(\mathbb{C})$ for $0 \leq j \leq N$; put $p = N/2$. Then we have*

$$[w_1\,\partial^p, \ldots, w_N\,\partial^p] = W^{0,1,\ldots,N-1}(w_1, \ldots, w_N) \cdot \partial^p,$$

where $W^{0,1,\ldots,N-1} = \mathbf{1} \wedge \partial \wedge \ldots \wedge \partial^{N-1}$ is the Wronskian determinant.

Proof. Permutations of arguments in the r.h.s. of (24) are reduced to permutations of w_js since $k_j \equiv p$. Let $\sigma \in S_N$ be a permutation and $\vec{\jmath} \in \mathbb{Z}^N \cap [0, Np]^N$ be a vector in the integral lattice. Suppose that the r.h.s. in (24) is expanded from left to right and all possible derivation combinations

$$S_{\sigma,\vec{\jmath}} \stackrel{\mathrm{def}}{=} (-1)^\sigma\, \partial^{j_1}(w_{\sigma(1)}) \cdot \ldots \cdot \partial^{j_N}(w_{\sigma(N)})$$

are obtained; we note that *not* all vectors $\vec{\jmath} \in \mathbb{Z}^N \cap [0, Np]^N$ can be realized: at least, $|\vec{\jmath}| \leq Np$. Still, the set $J = \{\vec{\jmath}\} \subset \mathbb{Z}^N \cap [0, Np]^N$ does not depend on σ. Assume there is a summand such that two functions w_a and w_b acquire equal numbers of derivations for some combination $\vec{\jmath} \in J$. Then, for the same combination $\vec{\jmath}$ and the transposition τ_{ab}, there is the permutation $\tau_{ab} \circ \sigma$ such that the order of w_a and w_b is reversed and $S_{\sigma,\vec{\jmath}} + S_{\tau_{ab}\circ\sigma,\vec{\jmath}} = 0$. Consequently, only the Wronskian remains at ∂^p owing to Eq. (25). □

Theorem 7 is a generalization of a perfectly familiar fact: the commutator of two vector fields is a vector field. We emphasize that Theorem 7 forbids the naive approach that combines N vector fields (e.g., symmetries of a PDE) in an attempt to obtain some vector field again.

Remark 3. Unfortunately, for arbitrary operators (23) of order $p = N/2$ this mechanism of compensations does not work. Indeed, suppose that the powers $k_j \leq p$ are arbitrary; then the sets $J \subset \mathbb{Z}^N \cap [0, Np]^N$ do *depend on* σ, and generally $J(\sigma) \neq J(\tau_{ab} \circ \sigma)$ if two functions w_a and w_b are differentiated w.r.t. z equal number of times in a summand $S_{\sigma,\vec{\jmath}(\sigma)}$. Of course, we can obtain the Wronskian determinant at some suitable power of ∂, but there can be much more summands, even at ∂^ℓ for $\ell \geq p$. The same difficulty occurs for the lower bound $k_j \geq p$, when we consider formal differential operators $\nabla = \sum_{j \geqslant p} w_j(z) \cdot \partial^j$.

Nevertheless, for an arbitrary integer $p' \geq (N-1)/2$ we have

$$[w_1\,\partial^{p'},\dots,w_N\,\partial^{p'}] = W^{0,1,\dots,N-1}(w_1,\dots,w_N)\cdot\partial^{Np'-N(N-1)/2}.$$

For various pairs $(N,p)\in\mathbb{N}\times\mathbb{N}$, one can deduce many extravagant phenomena. In [6], the following proposition is proved: for $p=1$, vector fields $\mathrm{D}(M^n)$ on a smooth n-dimensional manifold M^n are closed w.r.t N-ary bracket (24) and form the homotopy N-Lie algebra if $N=n^2+2n-2$.

As a corollary to Proposition 5, we have

Theorem 8. *Let N be even; consider the $\mathcal{O}(\mathbb{C})$-module $W_{N/2}\overset{\mathrm{def}}{=}\mathrm{span}_{\mathbb{C}}\langle w(z)\,\partial^{N/2}\rangle$ of holomorphic operators of order $N/2$. Then $W_{N/2}$ is endowed with the homotopy N-Lie algebra structure w.r.t. bracket* (24).

Nevertheless, the difficulties in complete description of the r.h.s. in (24) do not influence upon our ability to observe the homotopy N-Lie structure on the associative algebra of operators (23). As a reformulation to Proposition 5 on page 8 we obtain

Theorem 9. *Let N be even, then differential operators* (23) *of arbitrary orders compose the homotopy N-Lie algebra w.r.t. bracket* (24).

Indeed, the differential operators generate the associative algebra $\mathrm{Diff}_*(\mathcal{O}(\mathbb{C}),\mathcal{O}(\mathbb{C}))$.

2.2. **On the Wronskian determinants.** We start with

Proposition 10 ([5])**.** *Let k and l be positive integers, then the identity*

$$W^{0,1,\dots,k}[W^{0,1,\dots,l}]=0$$

holds.

Remark 4. Actually, a slight modification of Theorem 8 combined with Proposition 6 give a nice and compact proof of Proposition 10 in the case when the numbers k and l of the arguments are even. In the next section, we generalize Proposition 10 to the Wronskians $D^{\vec{\sigma}}$ w.r.t. several independent variables x^1, …, x^n, and, in particular, obtain its proof for arbitrary naturals k and l in the case $n=1$.

From Proposition 10 we also obtain

Theorem 11. *Let k and l be positive integers, then the relation*

$$[\![W^{0,1,\dots,k},W^{0,1,\dots,l}]\!]^{\mathrm{RN}}=0$$

holds.

Corollary 12. The d_W-cohomologies of the space of Wronskians are isomorphic to this space itself: $H^*_{\mathrm{d}_W}=\mathrm{span}_{\Bbbk}\langle W^{0,\dots,l},\ l\geq 1\rangle$, since the differential $\mathrm{d}_{W^{0,\dots,k}}=[\![W^{0,\dots,k},\cdot]\!]^{\mathrm{RN}}$ is trivial.

Now we study the homotopy generalizations of the Witt algebra (the Virasoro algebra with zero central charge) which is defined by the relations $[a_i,a_j]=(j-i)\,a_{i+j}$. Taking into account all our observations on the Wronskians, we set $N=2$ and consider the polynomial generators $a_i=x^{i+1}$, where $x\in\Bbbk$ and $i\in\mathbb{Z}$. For $N\geq 2$ and the Wronskian determinant $W^{0,1,\dots,N-1}$, we consider the relations

$$[a_{i_1},\dots,a_{i_N}]=\Omega(i_1,\dots,i_N)\,a_{i_1+\dots+i_N}, \tag{26}$$

where the structural constants $\Omega(i_1,\ldots,i_N)$ are skew-symmetric w.r.t. their arguments. Here we use the representation $a_i = x^{i+N/2}$. We claim that the function Ω is the Vandermonde determinant.

Theorem 13. *Let ν_1, …, $\nu_N \in \Bbbk$ be constants and set $\nu = \sum_{i=1}^N \nu_i$; then the equality*

$$W^{0,1,\ldots,N-1}(x^{\nu_1},\ldots,x^{\nu_N}) = \prod_{1\leq i<j\leq N}(\nu_j-\nu_i)\cdot x^{\nu-N(N-1)/2} \tag{27}$$

holds, i.e., the Wronskian determinant of the monomials is a monomial itself and its coefficient is the Vandermonde determinant.

Proof. Consider the determinant (27): $A = \det\|a_{ij}\,x^{\nu_j-i+1}\|$. From jth column take the monomial x^{ν_j-N+1} outside the determinant:

$$A = x^{\nu-N(N-1)}\cdot\det\|a_{ij}\,x^{N-i}\|;$$

all rows acquire common degrees in x: deg(any element in ith row) $= N-i$. From ith row take this common factor x^{N-i} outside the determinant:

$$A = x^{\nu-N(N-1)/2}\cdot\det\|a_{ij}\|,$$

where the coefficients a_{ij} originate from the initial derivations in a very special way: for any i such that $2\leq i\leq N$, we have

$$a_{1j}=1 \qquad \text{and} \qquad a_{ij} = (\nu_j\underline{-i+2})\cdot a_{i-1,j} \quad \text{for } 1<i\leq N.$$

The underlined summand does not depend on j, and hence for any $k=N$, …, 2 the determinant $\det\|a_{ij}\|$ can be splitted in the sum:

$$\det\|a_{ij}\| = \det\|a'_{kj} = \nu_j\cdot a_{k-1,j};\quad a'_{ij}=a_{ij} \text{ if } i\neq k\| + \\ + \det\|a''_{kj} = (2-i)\cdot a_{k-1,j};\quad a''_{ij}=a_{ij} \text{ if } i\neq k\|,$$

where the last determinant is trivial.

Solving the recurrence relation, we obtain

$$\det\|a_{ij}\| = \det\|\nu_j^{i-1}\| = \prod_{1\leq k<l\leq N}(\nu_l-\nu_k).$$

This completes the proof. □

Remark 5. We have calculated the structural constants in (26) by using another basis $a'_i = x^i$ such that the resulting degree is not $\sum_{k=1}^N \deg a'_k$. Nevertheless, the result is correct since we use the translation invariance of the Vandermonde determinant:

$$\Omega(i_1,\ldots,i_N) = \Omega(i_1+\frac{N}{2},\ldots,i_N+\frac{N}{2}),$$

and therefore all reasonings are preserved.

Now we recall the behaviour of bracket (13) w.r.t. a change of coordinates $y=y(x)$.

Theorem 14. *Let $\phi^i(y)$ be smooth functions for $1 \leqslant i \leqslant N$, i.e., ϕ^i is a scalar field of the conformal weight 0, such that ϕ^i is transformed by the rule $\phi^i(y) \mapsto \phi^i(y(x))$ under a change $y = y(x)$. Then the relation*

$$\det \left\| \frac{d^j \phi^i}{dx^j} \right\| \begin{array}{rcl} i & = & 1, \ldots, N \\ j & = & 0, \ldots, N-1 \\ \phi^i & = & \phi^i(y(x)) \end{array} = \left(\frac{dy}{dx} \right)^{\Delta(N)} \det \left\| \frac{d^j \phi^i}{dy^j} \right\| \begin{array}{rcl} \phi^i & = & \phi^i(y) \\ y & = & y(x) \end{array}$$

holds, where the conformal weight $\Delta(N)$ for the Wronskian determinant of N scalar fields ϕ^i of weight 0 is $\Delta(N) = N(N-1)/2$.

Proof. Consider a function $\phi^i(y(x))$ and apply the total derivative D_x^j by using the chain rule. The result is

$$\frac{d^j \phi^i}{dy^j} \cdot \left(\frac{dy}{dx} \right)^j + \text{terms of lower order derivatives } \frac{d^{j'}}{dy^{j'}}, \quad j' < j.$$

These lower order terms differ from the leading terms in $D_x^{j'} \phi^i(y(x))$, $0 \leq j' < j$, by the factors common for all i and thus they produce no effect since a determinant with coinciding (or proportional) lines equals zero. From ith row of the Wronskian we extract $(i-1)$th power of dy/dx, their total number being $N(N-1)/2$. This is the conformal weight by definition. □

We see that the Wronskian determinant of N functions is *not* a function itself: the objects we are dealing with are the higher order differential operators, and the functions are their coefficients w.r.t. the basis $\langle \mathbf{1}, \partial, \ldots \rangle$.

Theorem 14 can be extended to the case $n \geq 1$: $\boldsymbol{x} = (x^1, \ldots, x^n)$. We also see that the statement is generally not true if the generalized Wronskian is $\partial^{\sigma_1} \wedge \ldots \wedge \partial^{\sigma_N} \neq \text{const} \cdot \mathbf{1} \wedge \partial \wedge \ldots \wedge \partial^{N-1}$.

2.3. **Applications in the W–geometry.** We note that the concept of the homotopy Lie structures for differential operators (23) has a nice application in the W-geometry. The A_ℓ-W-geometry [9]–[11] is the geometry of complex curves Σ: $\dim_{\mathbb{C}}(\Sigma) = 1$, $\dim_{\overline{\mathbb{C}}}(\Sigma) = 1$, chirally embedded into the Kähler manifold $\mathbb{CP}^\ell$; further on, $f^A(z)$ and $\bar{f}^{\bar{A}}(\bar{z})$ are the embedding functions, $0 \leq A \leq \ell$ and $0 \leq \bar{A} \leq \ell$. The compatibility conditions of these embeddings are the Toda equations ([21]), associated with the semisimple A_ℓ-type Lie algebras.

Definition 4 ([10, §3.2])**.** A general infinitesimal W–*transformation* δ_W of such a curve is a change of the embedding functions f^A, $\bar{f}^{\bar{A}}$ of the form

$$\delta_W f^A(z) = \sum_{j=0}^{k} w_j(z)\, \partial^j f^A(z), \qquad \delta_W \bar{f}^{\bar{A}}(\bar{z}) = \sum_{j=0}^{k} \bar{w}_j(\bar{z})\, \bar{\partial}^j \bar{f}^{\bar{A}}(\bar{z}), \tag{28}$$

where $w_j \in \mathcal{O}(\mathbb{C})$ and $\bar{w}_j$are holomorphic and anti-holomorphic functions, respectively.

We see that a W-transformation is uniquely defined by the higher order differential operator $\sum_j w_j(z) \cdot \partial^j \in \mathrm{Diff}_*(\mathcal{O}(\mathbb{C}), \mathcal{O}(\mathbb{C}))$. So, by using Theorem 9 we conclude that the higher-order W-transformations compose the homotopy N-Lie algebras for even natural Ns.

3. The Wronskians in multidimensions: $n \geq 1$

In this section, we generalize the concept of the Wronskian determinants to the multidimensional case of the base $\boldsymbol{x} \in \mathbb{R}^n$; here we assume $\Bbbk = \mathbb{R}$. Further on, we consider the kth order jets $J^k(n,1)$ over the bundle $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, where the base is the Euclidean space of dimension $n \geq 1$ and the algebra $\mathcal{A}$ is the associative commutative algebra $C^\infty(\mathbb{R}^n)$ of smooth functions.

In order to construct a natural n-dimensional base generalization of the Wronskians, we pass to the geometrical standpoint ([3]) and make an experimental observation first.

Let $\mathcal{F}(\pi)$ be the algebra of smooth functions $C^\infty(J^\infty(\pi))$ and consider the $\mathcal{F}(\pi)$-module $\varkappa(\pi)$ of evolutionary vector fields[2] $\mathfrak{Э}_a = \sum_{j,\sigma} D_\sigma(a^j)\cdot\partial/\partial u^j_\sigma$, where $a^j \in \mathcal{F}(\pi)$. To each Cartan N-form $\omega \in \mathcal{C}^N\Lambda(\pi)$ we assign the operator $\nabla_\omega \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(N)}(\varkappa(\pi),\,\mathcal{F}(\pi))$ by the rule

$$\nabla_\omega(a_1,\ldots,a_N) = \mathfrak{Э}_{a_N} \lrcorner\, (\ldots(\mathfrak{Э}_{a_1} \lrcorner\, \omega)\ldots), \tag{29}$$

where $a_i \in \varkappa(\pi)$.

Proposition 15 ([3, Chapter 5])**.** *Correspondence* (29) *is the isomorphism of the* $\mathcal{F}(\pi)$-*modules*:

$$\mathcal{C}^N\Lambda(\pi) \simeq \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(N)}(\varkappa(\pi),\mathcal{F}(\pi)).$$

Further on, we use the notation $\omega(\mathfrak{Э}_{a_1},\ldots,\mathfrak{Э}_{a_N}) \stackrel{\mathrm{def}}{=} \mathfrak{Э}_{a_N} \lrcorner\, (\ldots(\mathfrak{Э}_{a_1} \lrcorner\, \omega)\ldots)$, where $\omega \in \mathcal{C}^N\Lambda(\pi)$ and $a_i \in \varkappa(\pi)$.

Remark 6 ([18]). Consider the infinite jets $J^\infty(\pi)$ over the bundle $\pi\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. Let $x \in \mathbb{R}$ be the independent base variable, u be the dependent fiber variable, D_x be the total derivative w.r.t. x, and $u^{(k)} \equiv D_x^k\, u$ be the coordinates in $J^\infty(\pi)$ for any $k \geq 0$. By $\mathrm{d}_\mathcal{C}$ we denote the Cartan differential, $\mathrm{d}_\mathcal{C}\colon C^\infty(J^\infty(\pi)) \to \mathcal{C}\Lambda(J^\infty(\pi))$, that maps $u^{(k)} \mapsto \mathrm{d}u^{(k)} - D_x u^{(k)}\,dx$. The Wronskian determinants (13) can be interpreted as action of the N-forms $\mathrm{d}_\mathcal{C}u \wedge \ldots \wedge \mathrm{d}_\mathcal{C}u^{(N-1)} \in \mathcal{C}\Lambda^*(J^{N-1}(\pi)) \subset \mathcal{C}\Lambda^*(J^\infty(\pi))$ upon the evolutionary vector fields $\mathfrak{Э}_{a_j} \equiv \sum_{k=0}^\infty D_x^k(a_j)\,\partial/\partial u^{(k)}$:

$$\begin{aligned}
[a_1,a_2] &= \mathrm{d}u \wedge \mathrm{d}(u')\,(\mathfrak{Э}_{a_1},\mathfrak{Э}_{a_2}),\\
[a_1,a_2,a_3] &= \mathrm{d}u \wedge \mathrm{d}(u') \wedge \mathrm{d}(u'')\,(\mathfrak{Э}_{a_1},\mathfrak{Э}_{a_2},\mathfrak{Э}_{a_3}), \qquad \text{etc.}
\end{aligned}$$

for any $a_j \in C^\infty(\mathbb{R})$. We emphasize that $a_j \in C^\infty(\mathbb{R}) \subset \varkappa(\pi)$, i.e., we restrict ourselves to a submodule of $\varkappa(\pi)$ generated by functions on the base M.

Remark 7. Consider the ternary bracket $\square \in \mathrm{Hom}(\bigwedge^3 C^\infty(\mathbb{R}^2), C^\infty(\mathbb{R}^2))$:

$$\square(a_1\wedge a_2\wedge a_3) = \mathrm{d}_\mathcal{C}u \wedge \mathrm{d}_\mathcal{C}u_x \wedge \mathrm{d}_\mathcal{C}u_y(\mathfrak{Э}_{a_1},\mathfrak{Э}_{a_2},\mathfrak{Э}_{a_3}) = \begin{vmatrix} a_1 & a_2 & a_3\\ D_x(a_1) & D_x(a_2) & D_x(a_3)\\ D_y(a_1) & D_y(a_2) & D_y(a_3)\end{vmatrix}.$$

For the bracket $\square$, the homotopy ternary Jacobi identity $\square[\square] = 0$ of the form (5) holds. We prove this fact by direct calculations using the Jet software [22].

[2]To denote evolutionary vector fields, we use the Cyrillic letter Э, which is pronounced like "e" in "ten".

Proposition 16 ([3]). *The dimension of the vertical part $J^k(n,1)/\mathbb{R}^n$ of the jet space $J^k(n,1)$ is*

$$\dim \frac{J^k(n,1)}{\mathbb{R}^n} = \dim J^k(n,1) - n = \sum_{i=0}^{k} \binom{n+i-1}{n-1} = \binom{n+k}{n}. \tag{30}$$

We also note that this dimension $N \equiv \binom{n+k}{n}$ is such that the inequality

$$\dim J^{k_1+k_2}(n,1) - n - 1 \geq \dim J^{k_1}(n,1) + \dim J^{k_2}(n,1) - 2(n+1)$$

is valid for any k_1 and k_2; in what follows, we need to substract the dimension $\dim J^0(n,1) = n+1$ in order to deal with non-trivial multiindexes σ such that $|\sigma| > 0$.

Choose arbitrary positive integers n and k; then $N = \binom{n+k}{n}$ is the dimension $\dim(J^k(n,1)/\mathbb{R}^n)$. Let $\mathcal{A} = C^\infty(\mathbb{R}^n)$ be the algebra of smooth functions $a_j \in \mathcal{A}$, $1 \leq j \leq N$. Now we define the N-linear skew-symmetric bracket $\square_k \in \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$: we put

$$\square_k(a_1, \ldots, a_N) = \bigwedge_{l=0}^{k} \Big(\bigwedge_{|\sigma|=l} \mathrm{d}_{\mathcal{C}} \cdot D_\sigma u \Big) (\eth_{a_1}, \ldots, \eth_{a_N}). \tag{31}$$

In coordinates, this bracket is $\square_k(a_1, \ldots, a_N) = \det \|D_{\sigma_i}(a_j)\|$, where $\sigma_i = (\sigma_1^i, \ldots, \sigma_n^i)$ runs through all multi-indexes such that u_{σ_i} is a coordinate on the kth jet space $J^k(n,1)$ of the fibre bundle $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.

We claim that the N-linear skew-symmetric bracket $\square_k \in \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ defined in (31) satisfies the homotopy N-Lie Jacobi identity

$$\square_k[\square_k] = 0. \tag{32}$$

To prove that, we establish a substantially more general

Theorem 17. *Let $N_{\mathrm{in}} = \binom{n+k_{\mathrm{in}}}{n}$ and $N_{\mathrm{out}} = \binom{n+k_{\mathrm{out}}}{n}$ be the dimensions given by Eq.* (30) *for some natural k_{in} and k_{out}; by $\square_{\mathrm{in}}$ and $\square_{\mathrm{out}}$ denote the multilinear skew–symmetric brackets defined in Eq.* (31)*. Then the equality*

$$\square_{k_{\mathrm{out}}}[\square_{k_{\mathrm{in}}}] = 0$$

holds.

Proof. Without loss of generality we assume that $k_{\mathrm{in}} \geq k_{\mathrm{out}}$, otherwise one has to transpose the subscripts 'in' and 'out' in Eq. (33).

In contrast with the reasoning in Section 2.2, we deal with $D^{\vec{\sigma}} = D_{\sigma_1} \wedge \ldots \wedge D_{\sigma_N}$, where σ_j is a multiindex $(\sharp x^1, \ldots, \sharp x^n) \in \mathbb{Z}_+^n$ for any j, $1 \leq j \leq N = \binom{n+k}{n}$. Define the norm $|D^{\vec{\sigma}}| = |\vec{\sigma}| = \sum_{j=1}^N |\sigma_j|$; we see that $|\square_{k_{\mathrm{out}}}[\square_{k_{\mathrm{in}}}]| = |\square_{k_{\mathrm{in}}}| + |\square_{k_{\mathrm{out}}}|$.

Now we note that the non-trivial skew-symmetric $(N_{\mathrm{in}} + N_{\mathrm{out}} - 1)$-linear bracket $\square_{\min} \in \mathrm{Hom}(\bigwedge^{N_{\mathrm{in}}+N_{\mathrm{out}}-1} \mathcal{A}, \mathcal{A})$ with the minimal norm is

$$\square_{\min} = \square_{k_{\mathrm{in}}} \wedge \Big(\sum_{\bar{\jmath} \in \Lambda^{N_{\mathrm{out}}-1}\left(J^{k_{\mathrm{in}}+k_{\mathrm{out}}}(n,1)/J^{k_{\mathrm{in}}}(n,1)\right)} \mathrm{const}(\bar{\jmath}) \cdot D^{\sigma_{\bar{\jmath}}} \Big), \tag{33}$$

where $\mathrm{const}(\bar{\jmath}) \in \mathbb{R}$ are some constant coefficients.

We claim that $|\square_{\min}| > |\square_{k_{\mathrm{in}}} + \square_{k_{\mathrm{out}}}|$, and thence $\square_{k_{\mathrm{out}}}[\square_{k_{\mathrm{in}}}] = 0$. Indeed, consider the r.h.s. in (33) and note that $|\Delta \wedge \nabla| = |\Delta| + |\nabla|$, see page 3 for definition of the

wedge product $\wedge$ in this case. The set of $N_{\rm out}$ different derivatives in $\Box_{k_{\rm out}}$ admits the canonical splitting:

$$\Box_{k_{\rm out}} = D^{\vec{\tau}^{\rm out}} = \mathbf{1} \wedge \underbrace{D_{\tau_2^{\rm out}} \wedge \ldots \wedge D_{\tau_{N_{\rm out}}^{\rm out}}}_{N_{\rm out}-1 \text{ factors}},$$

where $\vec{\tau}^{\rm out}$ contains all multiindexes in $J^{k_{\rm out}}(n,1)$, and those underbraced derivatives are in bijective correspondence with $N_{\rm out}-1$ different derivatives within any summand in the second wedge factor of (33) (there is the correspondence owing to the equal numbers of elements). Still,

$$1 \leq |D_{\tau_i^{\rm out}}| = |\tau_i^{\rm out}| \leq k_{\rm out} < k_{\rm in}+1 \leq |\sigma_{\bar{\jmath},i}| = |D^{\sigma_{\bar{\jmath},i}}| \leq k_{\rm in}+k_{\rm out} \qquad \forall i \neq 1, \quad \forall \bar{\jmath}.$$

Indeed, if a multiindex $\sigma_{\bar{\jmath},i}$ is such that $u_{\sigma_{\bar{\jmath},i}}$ is a coordinate on the jet space's part $J^{k_{\rm in}+k_{\rm out}}(n,1)/J^{k_{\rm in}}(n,1)$ with the higher order derivatives only, then $\sigma_{\bar{\jmath},i}$ is *longer* than any multiindex $\tau_i^{\rm out}$ such that $u_{\tau_i^{\rm out}}$ is a coordinate on $J^{k_{\rm out}}(n,1)$. Consequently, the norm of the second wedge factor in the r.h.s. of (33) is strictly greater than $|\Box_{k_{\rm out}}|$, and thence $\Box_{k_{\rm out}}[\Box_{k_{\rm in}}]$ is trivial. This completes the proof. $\Box$

Remark 8. The parity of the number $N = \binom{n+k}{n}$ of arguments in (31) is arbitrary and hence the reasonings of Theorem 17 exceed case (24) of the associative algebras; in particular, for $n=1$ we get Proposition 10, as we claimed in Remark 4.

We give an example of the homotopy 3-Lie algebra of polynomials in two variables:

Example 5. The space of polynomials $\mathrm{span}_{\Bbbk}\langle 1,x,y,xy\rangle \subset \Bbbk_2[x,y]$ endowed with the ternary bracket $\mathbf{1}\wedge D_x \wedge D_y$ acquires a homotopy 3-Lie algebra structure. The commutation relations in this algebra are

$$[1,x,y]=1, \qquad [1,x,xy]=x, \qquad [1,y,xy]=-y, \quad \text{and} \quad [x,y,xy]=-xy,$$

and we see that the structural constants are such that the generators x and y are mixed.

In this section, we have realized the continualization scheme: the N-ary bracket $W^{0,1,\ldots,N-1}$ is defined on the sequence of $\Bbbk$-algebras

$$\Bbbk_N[x] \hookrightarrow \Bbbk[[x]] \hookrightarrow \left[\begin{array}{c} \Bbbk[[x^1,\ldots,x^n]] \\ \{\sum_\alpha c_\alpha \cdot x^\alpha \mid \alpha \in \Bbbk,\ c_\alpha \in \Bbbk\}. \end{array}\right.$$

The sets of indexes are finite, cardinal, cardinal w.r.t. any of n generators, and continuous, respectively, for any possible N.

We also note that the definition of the Koszul ∂_Δ-cohomologies is invariant w.r.t. the number of derivations $\partial_i \colon \mathcal{A} \to \mathcal{A}$, $i=1,\ldots,n$, so that the cohomological constructions are preserved for the Wronskian determinant in (31) if $n \geqslant 1$. These concepts allow further, purely algebraic studies on the topic. Also, we note that there is a famous mechanism that provides the associative algebra structures, namely, the Yang-Baxter equation ([14]) and the WDVV equation (see [12] and references therein).

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