

# Decoherence suppression for oscillator-assisted geometric quantum gates via symmetrization

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We propose a novel symmetrization procedure to beat decoherence for oscillator-assisted quantum gate operations. The enacted symmetry is related to the global geometric features of qubits transformation based on ancillary oscillator modes, e.g. phonons in an ion-trap system. It is shown that the devised multi-circuit symmetrized evolution endows the system with a two-fold resilience against decoherence: insensitivity to thermal fluctuations and quantum dissipation.

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A key requirement to implement quantum computation consists in the ability to perform, with high accuracy, some set of universal gate operations for quantum bits (qubits) [1]. In the real world, a quantum system is never isolated, and the decoherence due to the surrounding environment is regarded as the main obstacle to practical realization of quantum information processing. A variety of techniques, based on the underlying mechanism of symmetry, have been proposed to protect quantum information, including quantum error correcting codes [2], decoherence-free subspaces [3], bang-bang dynamical decoupling [4], and noiseless subsystems [5, 6].

Most of these strategies rely, in a way or the other, on symmetry properties of the dynamics of the plain interactions of quantum information manipulation [6]. For instance, in the proposed schemes for dynamical decoupling [4], the qubit-qubit coupling is often invariant with respect to the symmetrization procedure exploited for the elimination of the qubit-environment couplings. On the other hand, it has been recently argued that robust quantum computing might be also achieved by a class of geometric transformations [7]. In particular, geometric quantum manipulation seems to possess a certain intriguing resilience to classical noises, i.e. stochastic fluctuations of the control parameters, due to its inherent global feature of the operation [8].

In this paper, we shall contrive a novel symmetrization procedure to beat decoherence for a sort of geometric quantum gates based on the use of ancillary oscillator modes. For the typical ion-trap system, the global gate operation assisted by a phonon mode [9] and the related time-reversal symmetry will be exploited. At variance with the previous schemes, the symmetry involved in our proposal will not directly show up at the level of dynamical interactions, i.e. Hamiltonian operator, but will relate to the geometric nature of the overall, finite-time evolution, i.e. unitary operator. The use of this sort of global transformations to implement quantum gates, and their extra advantages, have been already discussed by several authors [10]. However, the symmetry associated with these geometric operators and its potential relevance to the struggle against decoherence was never revealed.

Let us focus, for the sake of concreteness, on quantum

information processing based on ion-trap systems. Quantum manipulation on this kind systems constitutes one of the most promising approaches to quantum computation and it has been intensively studied during the past years [10, 11, 12]. In detail, qubits in the system are stored in internal atomic states controlled by laser beams and a controlled interaction between them is obtained via the coupling with ion vibrational states. Although the coupling of the qubits with the phononic degrees of freedom will generally lead to dependence of the gate operation on the motional states, recent studies show that such a drawback can be avoided by resorting to a control process enacting a gate with global features [9, 10]. The main idea of the scheme can be illustrated as below. Suppose that, by appropriately choosing the laser intensities and detunings, the following coupling of two ion qubits with the oscillator degree of freedom can be obtained

$$H(t) = [\alpha(t)a + \alpha^*(t)a^\dagger]J_y, \quad (1)$$

where the collective Pauli operator  $J_y = \sigma_y^{(1)} + \sigma_y^{(2)}$  accounts for the internal states of the ion qubits and  $\alpha(t)$  characterizes the interaction of the qubits with the vibrational state. When  $\alpha(t)$  has the form of certain periodic function such that in the time interval  $\int_0^T \alpha(t)dt = 0$ , the corresponding evolution generates a global transformation,  $e^{-iA(T)J_y^2}$  with  $A(T) = i \int_0^T \alpha(t) \int_0^t \alpha^*(t')dt'dt$ , which acts *only* on the internal states of the ions. The simplest case is given by the periodic step pulses, which leads to the following gate transformation

$$U_C(T) = P_+^x(\tau)P_+^p(\tau)P_-^x(\tau)P_-^p(\tau) = e^{-i2\alpha_0^2\tau^2 J_y^2}, \quad (2)$$

where  $\alpha_0$  and  $\tau = \frac{1}{4}T$  stand for the pulse amplitude and length respectively;  $P_\pm^x(\tau) = e^{\pm i\sqrt{2}\alpha_0\tau x J_y}$  and  $P_\pm^p(\tau) = e^{\pm i\sqrt{2}\alpha_0\tau p J_y}$  describe the unitary transformations for corresponding pulse interactions. [We have used the notation  $x = \frac{1}{\sqrt{2}}(a + a^\dagger)$  and  $p = \frac{i}{\sqrt{2}}(a^\dagger - a)$ ]. Note that the transformation (2) contains no operators with a non-trivial action over the vibrational degrees of freedom. The advantage of this gate operation is evident: it is independent of the vibrational states, hence is insensitive to the ion temperature.

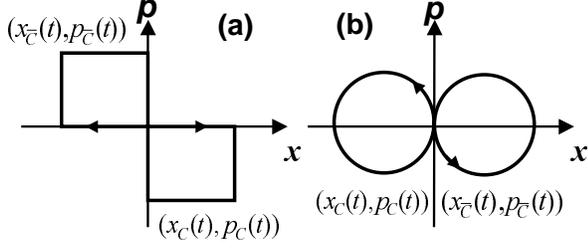


FIG. 1: The path traversed in phase space of the circuit evolution: (a) the case of step pulse interaction; (b) the circular case with  $\alpha(t) = \alpha_0 \cos \frac{\pi t}{2\tau} + i\alpha_0 \sin \frac{\pi t}{2\tau}$ .

One more feature of the above described operation, which will be at the very basis of the scheme to beat decoherence proposed in this paper, is the invariance of the global action under the *time reversal* of the interaction:  $H_C(t) \rightarrow H_{\bar{C}}(t) = -H_C(t)$ . This property can be easily checked in the case of the step pulse interaction. Indeed one has:  $U_{\bar{C}}(T) = P_-^x(\tau)P_-^p(\tau)P_+^x(\tau)P_+^p(\tau) = U_C(T)$ . Represented in Fig. 1 are the corresponding circuits in the oscillator phase space, including the above step pulse interaction and also the circular case with  $\alpha(t) = \alpha_0 \cos \frac{\pi t}{2\tau} + i\alpha_0 \sin \frac{\pi t}{2\tau}$ . The invariance of the global action under time reversal has, of course, a simple geometric interpretation: the corresponding paths traversed in the phase space, being related by the map  $\alpha \rightarrow -\alpha$ , indicate an anti-symmetric track geometry.

In real-life systems, the qubits and the oscillator degrees of freedom cannot be perfectly isolated. Here we shall focus on the case in which decoherence effects will arise by a thermal dissipation of the oscillator mode only. This latter will spoil the global action of (2) and induce qubit-oscillator entanglement which amounts to decoherence. The relaxation due to the coupling between the oscillator and the environment will be described by the following master equation

$$\begin{aligned} \partial_t \rho &= -i[H(t), \rho] + \frac{1}{2}\kappa(2a\rho a^\dagger - \{a^\dagger a, \rho\}) \\ &:= (\mathcal{L}_0 + \mathcal{L}_{th})\rho, \end{aligned} \quad (3)$$

where  $\kappa$  denotes the relaxation rate. The Liouville operator contains two terms  $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{th}$  with  $\mathcal{L}_0\rho = -i[H(t), \rho]$ . Let us first examine, for transparency, the influence of dissipation on the no-jump trajectory, namely, the evolution generated by the following non-Hermitian Hamiltonian

$$\tilde{H}(t) = H(t) - \frac{i}{2}\kappa a^\dagger a. \quad (4)$$

Let us consider the circuit of step pulses applied to the system. If the relaxation rate  $\kappa$  is small, i.e.  $\kappa \ll \alpha_0 \sim 1/\tau$ , then the non-unitary transformation generated by every pulse,  $\tilde{P}_\pm^{x,p}(\tau)$ , can be approximately expanded by resorting to the Baker-Hausdorff relation

$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} e^{-\frac{1}{12}([A,[A,B]] + [B,[B,A]])} \dots$ . In this way it is not difficult to realize that, to the first order in  $\kappa/\alpha_0$ , the sequence of four step pulses gives rise to the following transformation

$$\begin{aligned} \tilde{U}_C(T) &= \tilde{P}_+^x(\tau)\tilde{P}_+^p(\tau)\tilde{P}_-^x(\tau)\tilde{P}_-^p(\tau) \\ &\approx e^{-i2\alpha_0^2\tau^2 J_y^2} e^{-\sqrt{2}\kappa\alpha_0\tau^2 x J_y} e^{\sqrt{2}\kappa\alpha_0\tau^2 p J_y} e^{-2\kappa\tau a^\dagger a}. \end{aligned} \quad (5)$$

Clearly, the appearance of the middle two factors in the last term of Eq. (5) leads to the entanglement of qubits with the oscillator degrees of freedom, hence results in dependence of the transformation on the vibrational state.

The key observation is now to notice that the above fault can be effectively suppressed by performing a further time-reversed pulse sequence. One can readily obtain that the no-jump trajectory of the time-reversed pulse sequence will generate the transformation

$$\begin{aligned} \tilde{U}_{\bar{C}}(T) &= \tilde{P}_+^x(\tau)\tilde{P}_+^p(\tau)\tilde{P}_-^x(\tau)\tilde{P}_-^p(\tau) \\ &\approx e^{-i2\alpha_0^2\tau^2 J_y^2} e^{\sqrt{2}\kappa\alpha_0\tau^2 x J_y} e^{-\sqrt{2}\kappa\alpha_0\tau^2 p J_y} e^{-2\kappa\tau a^\dagger a}. \end{aligned} \quad (6)$$

From Eqs. (5) and (6), it can be seen that the influence of dissipation in the two circuits counteract each other. Therefore, a gate operation resistant against dissipation can be enacted by combining these two opposite pulse circuits

$$\begin{aligned} \tilde{U}(T_2) &= \tilde{U}_{\bar{C}}(T_2/2)\tilde{U}_C(T_2/2) \\ &\approx e^{-i2(\alpha_0\tau)^2 J_y^2} e^{-2\sqrt{2}\kappa\tau a^\dagger a}, \end{aligned} \quad (7)$$

where  $T_2 = \sqrt{2}T$  is the whole time period of the double-loop process. Since the two factors of the last term of Eq. (7) act individually on the qubits and the oscillator degree of freedom, the dissipation-induced dependence of the gate operation on the oscillator state has been removed.

It is important to stress that the cancellation of the unwanted influence of dissipation achieved by the above procedure is essentially a sort of group-symmetrization decoupling process [4]. Notably, in the present system, the enacted  $\mathbf{Z}_2$ -invariance of the dynamics under the map  $J_y \rightarrow -J_y$  resides in the geometric feature of the global transformation (2); therefore the devised symmetrized (anti-symmetrized) evolution (7) endows the scheme with a two-fold resilience against the noise sources due to the coupling with the oscillator: thermal fluctuations and dissipation.

The analysis carried over up to now was limited to the evolution in the no-jump trajectory. To get a full picture of the dissipative dynamics, i.e. the resolution of the master equation, generally requires to resort to a numerical approach. Remarkably, as we shall show in the following, the master equation of the present system (3) can be solved analytically by using group-theoretic techniques at a super-operatorial level.

Before presenting the solution, it is convenient to introduce the following notation [13]. Let us define the set of left operators  $O_L^i$  and right operators  $O_R^i$ : the left

operator action on the density matrix is given, as usual, by multiplication on left whereas the action of the right operators is defined as

$$O_R^1 \times O_R^2 \times \cdots \times O_R^m \rho = \rho O^m \times \cdots \times O^2 \times O^1. \quad (8)$$

By means of these notations, the operators contained in the master equation (3) can be conveniently described. For example, the operator  $\mathcal{L}_{th}$ , which describes the pure relaxation process, can be denoted by  $\mathcal{L}_{th} = \frac{1}{2}\kappa(2K_- - K_0 + 1)$ , where  $K_0 = a_L^\dagger a_L + a_R^\dagger a_R$ ,  $K_- = a_L a_R^\dagger$  and  $K_+ = a_L^\dagger a_R$ . Moreover, in view of the fact that these (super)operators generate a simple  $\text{su}(1,1)$  algebra:  $[K_0, K_\pm] = \pm 2K_\pm$ ,  $[K_-, K_+] = K_0$ , the process of pure relaxation is given by  $e^{\mathcal{L}_{th}t} = e^{-K_-} e^{\frac{\kappa t}{2}(1-K_0)} e^{K_-}$ . The established formalism lead in a rather straightforward fashion to the explicit description of the relaxation dynamics for a general oscillator coherent state  $|\beta\rangle$  [14],

$$e^{\mathcal{L}_{th}t} |\beta\rangle \langle \beta| = |\beta e^{-\frac{1}{2}\kappa t}\rangle \langle \beta e^{-\frac{1}{2}\kappa t}|. \quad (9)$$

Suppose now that the finite-time evolution governed by the master equation (3) is given by a map  $\Lambda$ , i.e.,

$$\rho(t) = \Lambda(t)\rho(0). \quad (10)$$

Note that a natural picture for the system is provided by basis of  $J_y$ -eigenstates  $|l\rangle$ . The time evolution for every element of the operator  $\Lambda(t)$ , defined as  $\Lambda^{ll'}(t) = \langle l | \Lambda(t) | l' \rangle$ , can be obtained:

$$\partial_t \Lambda^{ll'}(t) = (\mathcal{L}_0^{ll'} + \mathcal{L}_{th}) \Lambda^{ll'}(t), \quad (11)$$

where

$$\mathcal{L}_0^{ll'} = -il(\alpha a_L + \alpha^* a_L^\dagger) + il'(\alpha a_R + \alpha^* a_R^\dagger). \quad (12)$$

It should be clear that Eq. (11) suggests a linear algebraic structure, including, besides the aforementioned  $\text{su}(1,1)$  algebra, the both the left and right Heisenberg algebras  $\{a_L, a_L^\dagger, 1\}$  and  $\{a_R, a_R^\dagger, 1\}$ . According to the group-theoretic approach developed in Ref. [15], such a linear algebraic system is exactly solvable. We summarize below the solution of Eq. (11), details of the resolution will be reported elsewhere [14].

$$\begin{aligned} \Lambda^{ll'}(t) &= e^{-i\mathcal{A}^{ll'}(t)} e^{-il\xi_+^*(t)a_L^\dagger} e^{-i[l'\xi_+(t) + \Delta_{ll'}\xi_-(t)]a_L} \\ &\times e^{il'\xi_+(t)a_R} e^{i[l\xi_+^*(t) - \Delta_{ll'}\xi_-^*(t)]a_R^\dagger} e^{\mathcal{L}_{th}t}, \quad (13) \end{aligned}$$

where  $\Delta_{ll'} = l - l'$  and the coefficients are given by

$$\begin{aligned} \xi_\pm(t) &= e^{\mp\frac{1}{2}\kappa t} \int_0^t \alpha(t') e^{\pm\frac{1}{2}\kappa t'} dt', \\ \mathcal{A}^{ll'}(t) &= i\kappa ll' \int_0^t |\xi_+(t')|^2 dt' \\ &\quad - i \int_0^t [l^2 \alpha(t') \xi_+^*(t') + l'^2 \alpha^*(t') \xi_+(t')] dt'. \quad (14) \end{aligned}$$

The above analytical solution fully characterizes the decoherence effects induced by the dissipation of the oscillator mode, e.g. vibrational relaxation in an ion trap. The integrals in the functions  $\xi_\pm(t)$  might clearly be non zero; therefore, the transformation described by equation (13) will be inevitably depend on the oscillator state. More specifically, let us suppose that the oscillator state is prepared in a general coherent state  $|\beta\rangle$  and the whole system is initially prepared in the product state  $\rho(0) = \rho_{in}(0) \otimes |\beta\rangle \langle \beta|$ . By substituting  $\Lambda^{ll'}(t)$  of (13) into equation (10) and tracing over the oscillator degree of freedom, one obtains the evolution of the reduced density matrix of the qubits

$$\begin{aligned} \rho_{in}^{ll'}(t) &= e^{-i\mathcal{A}^{ll'}(t)} e^{ll'|\xi_+(t)|^2} \\ &\times e^{-i\Delta_{ll'}(\beta\xi_-(t) + \beta^*\xi_-^*(t))} e^{-\frac{1}{2}\kappa t} \rho_{in}^{ll'}(0), \quad (15) \end{aligned}$$

where we have taken the element  $\rho_{in}^{ll'}(t)$  in the  $J_y$  representation.

It should now become evident that the dissipation-induced effects can be suppressed by the symmetrization procedure. The application of the time-reversed pulse circuit  $\alpha(t)$  pulse will quench the integrations contained in  $\xi_\pm(t)$ . Specifically, note that the polynomial expressions of  $\xi_\pm(t)$  is given by  $\xi_\pm(t) = e^{\mp\frac{1}{2}\kappa t} \sum_{k=0}^{\infty} \frac{(\pm)^k}{k!} I^{(k)}(t)$ , where  $I^{(k)}(t) = \int_0^t \alpha(t') (\frac{\kappa}{2}t')^k dt'$ . For the step pulse sequence indicated by Eq. (2), one has  $I^{(0)}(T) = 0$  and nonvanishing terms  $I^{(k)}(T) \propto (\kappa T)^k$  of  $k > 0$ ; but for the time-reversal symmetrized pulse sequence indicated by Eq. (7), there are  $I^{(0)}(T_2) = I^{(1)}(T_2) = 0$ , leaving only higher order terms of  $k > 1$ . In fact, such a symmetrization procedure amounts to a decoupling, in which the dissipation effects are removed to the lowest-order in the scale  $\kappa/\alpha_0$ . In principle, it is possible to cancel the decoherence effects to an arbitrary high order by an iterative application of the above symmetrization procedure. The  $k$ -order decoupling pulse sequence to implement the gate operation (2) is illustrated below

$$U^{(k)}(T_n) = \underbrace{\cdots \cdots U_{\bar{C}} U_C U_C U_{\bar{C}} U_C U_{\bar{C}} U_C U_{\bar{C}} U_C}_{(k)}, \quad (16)$$

(1)  
(2)

where  $n = 2^k$  is the number of the pulse circuits and  $T_n = \sqrt{n}T$  is the total time length of the above pulse sequence.

It is worthwhile to point out that besides inducing the mentioned dependence of the qubit evolution on the oscillator state, dissipation will also result in a deviation from the desired gate operation. This latter effect is indicated in the coefficient  $\mathcal{A}^{ll'}(t)$ . Note that this undesired influence of dissipation, being embedded within the geometric action, cannot be removed by the above symmetrization procedure. In Fig. 2 we report in detail the fidelities of the gate operation for different applications of the pulse

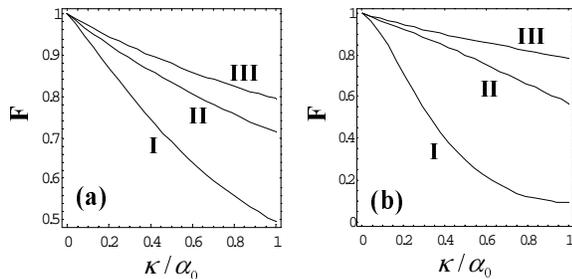


FIG. 2: The fidelity of the gate operation [Eq. (17)] for various applications of step pulse sequences. Curves I, II, and III correspond respectively to the symmetrization procedure with decoupling order  $k = 0, 1$  and  $2$ , in which the number of the pulse circuit is taken as  $n = 1, 2$  and  $4$  accordingly. The initial state of the oscillator degree of freedom assumes (a)  $\beta = 2$  and (b)  $\beta = 5$ .

sequences as a function of the ratio  $\kappa/\alpha_0$ . Fidelity is defined as

$$F = \langle \Psi_{\max} | \rho_{in}(T_n) | \Psi_{\max} \rangle, \quad (17)$$

in which  $T_n = \sqrt{n\pi}/\alpha_0$  with  $n$  the number of the circuits and  $|\Psi_{\max}\rangle = e^{-i\frac{\pi}{8}J_y^2}|00\rangle$  is the fully entangled state attained by the ideal gate operation acting on the qubit state  $|00\rangle$ .

In summary, we have contrived a novel symmetrization procedure for oscillator-assisted geometric quantum gates to suppress the detrimental effects induced by dissipation of the oscillator mode. The existence of the symmetry is related to the invariance of this class of geometric operations under the finite-time reversal transformation. We have devised a time-reversal symmetrization procedure which purports to eliminate quantum decoherence due to the oscillator dissipative relaxation to an arbitrary high order. The resulting quantum computation scheme possesses a two-fold resilient character against decoherence: insensitive to the oscillator thermal fluctuations and immune to quantum dissipation. Note that a similar symmetrization procedure was involved in a former refocusing scheme for holonomic quantum computation [16]. In comparison, the present work suggests a nonadiabatic manner to implement a geometrical gate operation robust against both classic and quantum noises. Moreover, the resistance against dissipation in the present scheme has been demonstrated by analytically solving the master equation by means of a powerful group-theoretical method at the superoperatorial level. On the practical side, we expect our results to be especially relevant for ion-trap quantum computing by geometrical gates.

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